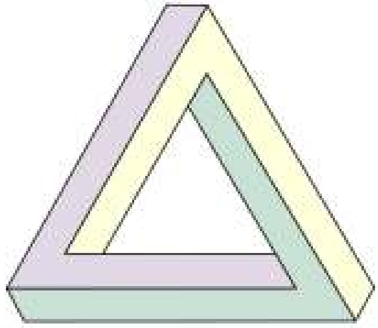
What are the shapes of the triangles?



(1) Given 12, 15, 20 be the sides of the triangle ΔABC. If we use the altitudes h_a, h_b and h_c of ΔABC as lengths to construct another ΔXYZ. What kind of triangle is ΔXYZ ?

Method 1

$$s = \frac{1}{2}(h_a + h_b + h_c) = \frac{1}{2}(12 + 15 + 20) = \frac{47}{2}$$

By Heron Formula, the area of ΔXYZ , S is

$$S = \sqrt{s(s-h_a)(s-h_b)(s-h_c)} = \sqrt{\frac{47}{2}\left(\frac{47}{2}-12\right)\left(\frac{47}{2}-15\right)\left(\frac{47}{2}-20\right)} = \frac{\sqrt{128639}}{4}$$

Hence $h_a = \frac{2S}{12} = \frac{\sqrt{128639}}{6}$, $h_b = \frac{2S}{15} = \frac{2\sqrt{128639}}{15}$, $h_c = \frac{2S}{20} = \frac{\sqrt{128639}}{10}$,

$$h_b{}^2 + h_c{}^2 = \left(\frac{2\sqrt{128639}}{15}\right)^2 + \left(\frac{\sqrt{128639}}{10}\right)^2 = \frac{128639}{36}$$

$$h_a{}^2 = \left(\frac{\sqrt{128639}}{6}\right)^2 = \frac{128639}{36}$$

$\therefore\ h_b{}^2 + h_c{}^2 = h_a{}^2$

By the Converse of Pythagoras Theorem, ΔXYZ is a right angled triangle.

Obviously ΔXYZ is scalene, since all three sides h_a, h_b and h_c are different.

The calculation of the exact length of the altitudes may be difficult. A better method is shown below.

Method 2

Let S be the area of the triangle, then $S = \frac{1}{2}h_a(12) = \frac{1}{2}h_b(15) = \frac{1}{2}h_c(20)$

$$h_a : h_b : h_c = \frac{1}{12} : \frac{1}{15} : \frac{1}{20} = \frac{60}{12} : \frac{60}{15} : \frac{60}{20} = 5:4:3 \implies h_a = 5k,\ h_b = 4k,\ h_c = 3k$$

Since $h_b{}^2 + h_c{}^2 = (4k)^2 + (3k)^2 = 25k^2 = h_a{}^2$

By the Converse of Pythagoras Theorem, ΔXYZ is a right angled triangle.

Obviously ΔXYZ is scalene, since all three sides $h_a : h_b : h_c = 5:4:3$ and are different.

(2) Given that a , b and c are the three sides of ΔABC,

If $c^2(a^2 + b^2 - c^2) = b^2(c^2 + a^2 - b^2)$, then what kind of triangle is ΔABC ?

$$c^2(a^2 + b^2 - c^2) = b^2(c^2 + a^2 - b^2)$$
$$c^2a^2 + b^2c^2 - c^4 - b^2c^2 - a^2b^2 + b^4 = 0$$

$$b^4 - c^4 - a^2(b^2 - c^2) = 0$$
$$(b^2 + c^2)(b^2 - c^2) - a^2(b^2 - c^2) = 0$$
$$(b^2 - c^2)(b^2 + c^2 - a^2) = 0$$

(i) If $b^2 - c^2 = 0$, $b^2 = c^2$, $b = c$, then ΔABC is isosceles.

(ii) If $b^2 + c^2 - a^2 = 0$, then $a^2 = b^2 + c^2$.

By the Converse of Pythagoras Theorem, ΔABC is a right angled triangle with $\angle ABC = 90^o$.

(3) In ΔABC, if $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$, then what is the shape of the triangle ?

Method 1

Use Cosine Law and Sine Law for $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$

$$\frac{\frac{b^2+c^2-a^2}{2bc} + 2\frac{a^2+b^2-c^2}{2ab}}{\frac{b^2+c^2-a^2}{2bc} + 2\frac{a^2+c^2-b^2}{2ac}} = \frac{b}{c}$$

$$\frac{a(b^2+c^2-a^2) + 2c(a^2+b^2-c^2)}{a(b^2+c^2-a^2) + 2b(a^2+c^2-b^2)} = \frac{b}{c}$$

$$ac(b^2 + c^2 - a^2) + 2c^2(a^2 + b^2 - c^2) = ab(b^2 + c^2 - a^2) + 2b^2(a^2 + c^2 - b^2)$$
$$a(b^2 + c^2 - a^2)(c - b) + 2[a^2(c^2 - b^2) - (c^4 - b^4)] = 0$$
$$(c - b)[a(b^2 + c^2 - a^2) + 2(c + b)(a^2 - c^2 - b^2)] = 0$$
$$(c - b)(b^2 + c^2 - a^2)(a - 2b - 2c) = 0$$

Since by Triangular Inequality, $2b + 2c > 2a > a \Longrightarrow a - 2b - 2c \neq 0$.

$\therefore c = b$ or $b^2 + c^2 = a^2$

Therefore ΔABC is an isosceles triangle or a right-angled triangle with $\angle A$ as right-$\angle$.

Method 2

$$\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$$

$$\cos A \sin C + 2\sin C\cos C = \cos A \sin B + 2\sin B\cos B$$
$$\cos A \sin C + \sin 2C = \cos A \sin B + \sin 2B$$
$$\cos A\,(\sin C - \sin B) + (\sin 2C - \sin 2B) = 0$$

$$\cos A\,\left[2\cos\left(\frac{C+B}{2}\right)\sin\left(\frac{C-B}{2}\right)\right] - 2\cos\,(C + B)\sin(C - B) = 0$$

$$\cos A\,\left[2\cos\left(\frac{180° - A}{2}\right)\sin\left(\frac{C-B}{2}\right)\right] - 2\cos\,(180° - A)\sin(C - B) = 0$$

$$\cos A\left[2\sin\left(\frac{A}{2}\right)\sin\left(\frac{C-B}{2}\right)\right]+2\cos A\left[2\sin\left(\frac{C-B}{2}\right)\cos\left(\frac{C-B}{2}\right)\right]=0$$

$$2\cos A\ \sin\left(\frac{C-B}{2}\right)\left[\sin\left(\frac{A}{2}\right)+2\cos\left(\frac{C-B}{2}\right)\right]=0$$

$$\cos A=0 \text{ or } \sin\left(\frac{C-B}{2}\right)=0 \text{ or } \sin\left(\frac{A}{2}\right)+2\cos\left(\frac{C-B}{2}\right)=0$$

(i) If $\cos A=0$, then $\angle A=90°$, ΔABC is a right-angled triangle.

(ii) If $\sin\left(\frac{C-B}{2}\right)=0$, then $\frac{C-B}{2}=0$, $\angle C=\angle B$ and ΔABC is an isosceles triangle.

(iii) (a) $0°<\frac{A}{2}<90° \implies \sin\left(\frac{A}{2}\right)>0$

(b) $-90°<\frac{C-B}{2}<90° \implies 2\cos\left(\frac{C-B}{2}\right)>0$

(c) $\sin\left(\frac{A}{2}\right)+2\cos\left(\frac{C-B}{2}\right)\neq 0$

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